Real Numbers

EXERCISE 1.3

1) Prove that $\sqrt{5}$ is irrational:
   Answer:

   Let $\sqrt{5}$ be a rational number. Therefore, we can find two integers a,b (b ≠ 0) such that

   $\sqrt{5} = \frac{a}{b}$. Let a and b have a common factor other than 1. Then we can divide them by the common factor, and assume that a and b are co-prime.

$$a = \sqrt{5}\, b$$

$$\rightarrow a^2 = 5\, b^2$$

   Therefore, a^2 is divisible by 5 and it can be said that a is divisible by 5.

   Let a = 5k , where k is an integer

$$(5k)^2 = 5b^2$$

$$\rightarrow 5k^2 = b^2$$

   This means that b^2 is divisible by 5 and hence, b is divisible by 5.

   This implies that a and b have 5 as a common factor.

   And this is a contradiction to the fact that a and b are co-prime.

   Hence, $\sqrt{5}$ cannot be expressed as $\frac{p}{q}$ or it can be said that $\sqrt{5}$ is irrational.

2) Prove that $3 + 2\sqrt{5}$ is irrational.
   Answer:

   Let $3 + 2\sqrt{5}$ be rational. Therefore, we can find two co-prime integers a, b (b ≠ 0) such that

$$3 + 2\sqrt{5} = \frac{a}{b}$$

$$\Rightarrow\ 2\sqrt{5} = \frac{a}{b} - 3$$

$$\Rightarrow \sqrt{5} = \frac{1}{2}\left(\frac{a}{b} - 3\right)$$

   Since a and b are integers, $\frac{1}{2}\left(\frac{a}{b} - 3\right)$ will also be rational at 5 and therefore $\sqrt{5}$ is rational.This contradicts the fact that $\sqrt{5}$ is irrational. Hence, our assumption that $3 + 2\sqrt{5}$ is rational is false.Therefore $3 + 2\sqrt{5}$ is irrational.

3) Find the LCM and HCF of the following integers by applying the prime factorization method.

(i) $\frac{1}{\sqrt{2}}$ (ii) $7\sqrt{5}$ (iii) $6+\sqrt{2}$

Answer:

(i) $\frac{1}{\sqrt{2}}$

Let $\frac{1}{\sqrt{2}}$ is rational.

Therefore, we can find two co-prime integers a,b (b≠0) such that

$$\frac{1}{\sqrt{2}} = \frac{a}{b}$$

Or $$\sqrt{2} = \frac{a}{b}$$

$\frac{b}{a}$ is rational as a and b are integers.

Therefore, $\sqrt{2}$ is rational which contradicts the fact that $\sqrt{2}$ is irrational. Hence, our assumption is false and $\frac{1}{\sqrt{2}}$ is irrational.

(ii) $7\sqrt{5}$

Let $7\sqrt{5}$ be rational.

Therefore, we can find two co-prime integers a,b (b≠0) such that

$$7\sqrt{5} = \frac{a}{b}$$

Or $$\Rightarrow \sqrt{5} = \frac{a}{7b}$$

$\frac{a}{7b}$ is rational as a and b are integers.

Therefore, $\sqrt{5}$ is rational. This contradicts the fact that $\sqrt{5}$ is irrational. Therefore, our assumption is $7\sqrt{5}$ is rational is false. Hence, $7\sqrt{5}$ is irrational.

(iii) $6 + \sqrt{2}$

Let $6 + \sqrt{2}$ be rational.

Therefore, we can find two co-prime integers a,b (b≠0) such that

$$6 + \sqrt{2} = \frac{a}{b}$$

Or $$\Rightarrow \sqrt{2} = \frac{a}{b} - 6$$

Since a and b are integers, $\frac{a}{b} - 6$ is also rational and hence, $\sqrt{2}$ should be rational. This contradicts the fact that $\sqrt{2}$ is irrational. Therefore, our assumption is false and hence, $6 + \sqrt{2}$ is irrational.